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2-Adic Groups and Character Formulas

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INTRODUCTION

In this paper we study 2-adic groups. In the case where $\imath$ is in the Galois group, the algebra contains only a quaternion group or is split, and when $\imath$ is not in the Galois group, the algebra simply splits. In the end, this paper presents a character criterion for 2-adic splitting. Among the techniques we use are inflation (inf) and deflation (def) in order to determine the order of the commutator, and inf always so the commutator maps onto the old roots of unity. When this happens, we form a new factor set which commutes.

NOTATION

Consider a cyclic group C such that $|C| = u2^n$, where $2^{r-1} < u \leqslant 2^r - 1$. Let K be a field such that $K \supseteq Q_2$, the 2-adic rationals, $L = K(C)$, and $G = \mathrm{Gal}(L/K)$. We define an action of G on C by

$$(c, \sigma) \to c^{\sigma}.$$

This action of G is faithful; if $a^{\gamma} = a$ for a in C, then $\gamma = 1$. G is an abelian group acting on a cyclic group. We write a^{σ} for $a^{\sigma^{-1}}$.

Consider the exact sequence

$$0 \to C \to E \to G \to 0,$$

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where the set $E = C \times G$, and identify u_σ with $(1, \sigma)$. Define a factor set $\{\beta(\sigma, \tau)\}$ where $u_\sigma u_\tau = \beta(\sigma, \tau) u_{\sigma\tau}$ with $\beta(\sigma, \tau)$ in C. Note that $\mathrm{Gal}(L/Q_2) \supseteq \mathrm{Gal}(L/K)$ with generators ι, η, and θ of G acting on C_2, the Sylow 2-subgroup of C, by

$$x^\iota = x^{-1} \text{ for } x \text{ in } C_2 \quad \text{and} \quad y^\iota = y \text{ for } y \text{ in } C_{2'}, \text{ the } 2'\text{-subgroup of } C,$$

$$x^\theta = x^5 \text{ for } x \text{ in } C_2 \quad \text{and} \quad y^\theta = y \text{ for } y \text{ in } C_{2'},$$

$$x^\eta = x \text{ for } x \text{ in } C_2 \quad \text{and} \quad y^\eta = y^2 \text{ for } y \text{ in } C_{2'}.$$

The group $\mathrm{Gal}(L/Q_2) = \langle \iota \rangle \times \langle \eta \rangle \times \langle \theta \rangle$.

Define $\tau = \theta^{2^\lambda}$ such that θ^{2^λ} lies in $\mathrm{Gal}(L/K)$ and $\theta^{2^{\lambda-1}}$ does not. Now the Frobenius automorphism η of G may be chosen as 2^κ on $C_{2'}$ and sends z to z^{5^n} for z in C_2 where $n \neq 0$ and κ is the unramified degree of K over Q_2. Consider $C_2 = \langle x \rangle$. Define integers a, b, and c as follows:

$$u_\tau u_\eta = x^a u_\eta u_\tau,$$

$$u_\eta u_\iota = x^b u_\iota u_\eta,$$

$$u_\iota u_\tau = x^c u_\tau u_\iota.$$

By using inf so that $\mathrm{im}(\tau - 1) = C_2$, we may change the factor set to make $a = c = 0$. Also use inf to obtain $|C_{2'}| = 2^r - 1$.

Basics of Deflation

Let N be a subgroup of G. Then we have an exact sequence

$$0 \to C^N \to E^* \to G/N \to 0.$$

Theorem. *We have $C^N \cong C/[C, N]$ and $G/N = \langle \iota \rangle \times \langle \eta \rangle / \langle \eta^2 \rangle$.*

Proof. Under deflation $(\bar{x}^b)^2 = 1$.

Compute $\bar{u}_\eta(\bar{u}_\eta \bar{u}_\iota) = \bar{u}_\eta 2\bar{u}_\iota = (\bar{x}^b)^2 \bar{u}_\iota \bar{u}_\eta^2$.

Note that η^{-1} is also a fifth power, so the endomorphism $1 + \eta^{-1}$ has order 2. Let G_0 be the subgroup generated by C^N, u_ι, and u_η. Apply the same exact sequence to G_0 and obtain ker def $= 1$. Then obtain that $(x^b)^2 = 1$. Consider ιu_η. Then

$$[\iota u_\eta, u_\iota] = [\iota, u_\iota][u_\eta, u_\iota].$$

Consider $[\iota u_\eta, u_\tau] = \iota^{\tau - 1}[u_\eta, u_\tau] = 1$.

CHARACTER FORMULAS

Let H be the 2-subgroup of G and η an irreducible character of H. We have that $\eta(x) = \eta(x^{-1})$ are real, and, in fact, we have that $\beta(\iota, \iota) = 1$ or -1 by

$$(1/|H|) \sum_{h \text{ in } H} \eta(h^2).$$

Let χ be an irreducible character of G. Then $\chi|_H = r(\sum \eta^x)$, where all η^x are real and are conjugate.

THEOREM. *$\beta(\iota, \iota) = 1$ or -1.*

Depends on $\sum \chi|_H(h^2)$.

CHARACTER THEOREM. *Let χ be a character of the K-special group G.*

(a) *Let ι be in G.*

(i) *If $\sum_{h \text{ in } H} \chi(h^2) \geq 0$, then the 2-adic Schur index $m_2(\chi) = 1$.*

(ii) *If $\sum_{h \text{ in } H} \chi(h^2) < 0$ and $[K: Q_2]$ is odd, then $m_2(\chi) = 2$.*

(iii) *If $\sum_{h \text{ in } H} \chi(h^2) < 0$ and $[K: Q_2]$ is even, then $m_2(\chi) = 1$.*

(b) *If ι is not an element of G, then $m_2(\chi) = 1$.*

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